Math 206B Lecture 18 Notes

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1 Relationships of Schur Functions to Characters of $\mathrm{GL}_N(\mathbb{C})$ and to Young Tableau

1.1 Schur functions and characters of $\mathrm{GL}_N(\mathbb{C})$

Definition 1.1. The ring of **symmetric functions of degree** n is $\mathbb{C}[x_1, \dots, x_n]^{S_n}$. The ring of **symmetric functions** is $\Lambda = \varprojlim_n \Lambda_n$.

Example 1.1. Λ_3 is spanned by $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$ and $x_1x_2x_3$.

A representation is a homomorphism $\rho : V \ to \operatorname{Aut}(V)$, where $V = \mathbb{C}^d$. Essentially, what we do with representations of finite groups can be done with compact groups, as well.

Theorem 1.1. *All representations of $\mathrm{GL}_N(\mathbb{C})$ are rational. That is, if $M = (x_{i,j})$, then $\rho(M) = (f_{pq}(x_{i,}))$, where the $f_{p,q}$ are rational polynomials. Moreover, $\rho(M) = (\det)^k$ times a polynomial representation.*

Example 1.2. Let ρ be the determinant map sending $M \mapsto \det(M)$. This is a 1-dimensional representation. Then $\rho_k(M) = (\det(M))^k$ is also a representation for $k \in \mathbb{Z}$.

Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_N$ with $\lambda \in \mathbb{N}$. If ρ is an irreducible representation of $\mathrm{GL}_N(\mathbb{C})$, then consider diagonal matrices M and "characters of ρ" given by $\operatorname{tr}(\rho[M])$. Then these is in Λ_N.

Theorem 1.2 (Weyl[1])**.** *If π is an irreducible representation of $\mathrm{GL}_N(\mathbb{C})$ corresponding to $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$, then $\operatorname{tr}(\pi) = s_\lambda$, where $s_\lambda = a_{\lambda+\rho}/a_\rho$ is a Schur function.*

Here, if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $a_\alpha = \det(x_i^{\alpha_j})_{i,j=1,\dots,N}$ is like a Vandemonde determinant, and $\rho = (N-1, N-2, \dots, 0)$ (so a_ρ is a Vandermonde determinant).

Example 1.3. Let $N = 2$ and $\lambda = (4, 3)$. Then

$$s_\lambda = \frac{\det \begin{bmatrix} x^4 & y^4 \\ x^3 & y^3 \end{bmatrix}}{x - y} = \frac{x^4y^3 - y^4x^3}{x - y} = x^3y^3.$$

[1] Igor does not remember whose theorem this is, so Weyl is a guess.

Remark 1.1. The situation in the theorem is actually something that happens for all compact groups.

1.2 Formula for Schur functions in terms of Young tableau

Theorem 1.3. *Let* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$. *Then*

$$s_\lambda = \sum_{A \in \mathrm{SSYT}(\lambda, \leq N)} x_1^{m_1(A)} \cdots x_N^{m_N(A)},$$

where $m_i(A)$ *is the number of* i*s in* A.

Lemma 1.1. *This sum is a symmetric polynomial of degree* N.

Proof. We show that the sum is invariant under the transposition $(i\, i+1) \in S_N$ for all $i = 1, \ldots, N-1$. Given a semistandard Young tableau, look at each row. If we look at rows with i and $i+1$, look at parts where we do not have $i+1$ squares directly below i squares. Then we can switch the number of is and $(i+1)$s in this part of each row and still get a semistandard Young tableau. □

Proof. We can prove the theorem by showing that a_ρ times the sum is $a_{\lambda+\rho}$.

$$\prod_{1 \leq i < j \leq N} (x_i - x_j) = \sum_{\sigma \in S_N} \mathrm{sign}(\sigma) s^0_{\sigma(1)} x - \sigma(2)^1 \cdots x^{N-1}_{\sigma(N)}$$

Here is Gessel's proof of this fact. It is easier to consider $\prod_{i<j}(1 - x_j/x_i)$. □